A SHORT REVIEW of CAGD TOOLS for ISOGEOMETRIC ANALYSIS

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IsoGeometric Analysis: a New Paradigm for the Discretization of PDEs

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Isogeometric Analysis

- a method for the analysis of problems governed by PDE:
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  - more accurate modeling of complex geometries/exact representations of common engineering shapes

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  - simplified mesh refinement
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  - simplified mesh refinement
- “the solution space for dependent variables is represented in terms of the same functions which represent the geometry”
  [Hughes-Cottrel-Bazilevs, 2005]

↓

Isogeometric analysis
exact description of the geometry at any level
Isogeometric Analysis (based on NURBS)

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[J.A. Cottrell, T.J.R. Hughes, Y. Bazilevs, CMAME, 2005]
Isogeometric Analysis (based on NURBS)

- exact description of the geometry at any level
- several efficient possibilities of refinement
  - $h$– refinement (knot-insertion/knot-refinement)
  - $p$– refinement (degree-elevation/degree-raising)
  - $k$– refinement (suitable combinations of the previous)

In a sense, Isogeometric Analysis (based on NURBS) is a superset of FEA [Cottrel-Bazilevs-Beirao da Vega-Cottrell-Huges-Sangalli, 2006]
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$h$—refinement, [J.A. Cottrell, T.J.R. Hughes, Y. Bazilevs, CMAME, 2005]
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[J.A. Cottrell, T.J.R. Hughes, A. Reali, CMAME, 2007]
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[Cottrel-Bazilevs-Beirao da Vega-Cottrell-Huges-Sangalli, 2006]
Goal

- present basic algorithms/properties of B-splines of salient interest
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- propose a possible alternative to the rational (NURBS) model
Outline

B-splines
  - definition
  - main properties
  - main algorithms
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- B-splines → NURBS

- Alternatives to the rational model
B-splines

- B-spline functions/curves/surfaces are NOT special piecewise polynomial (p.p.) functions/curves/surfaces
B-splines

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- Why are B-splines so popular?
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- Why are B-splines so popular?
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Why are B-splines so popular?

- they are the **best** way to represent p.p. both from the geometric and computational point of view
- there exist **efficient** and **stable** algorithms for their evaluation/manipulation
Notation: control polygon

\[ \mathcal{U} := \langle \varphi_1(t), \ldots, \varphi_m(t) \rangle, \quad <1,t> \subseteq \mathcal{U} \]
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- curves in \( \mathbb{R}^d \)

\[ C(t) := \sum_{i=1}^{m} c_i \varphi_i(t), \]

\( c_i \in \mathbb{R}^d \) control points
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- scalar functions
\[
f(t) := \sum_{i=1}^{m} c_i \varphi_i(t),
\]

\[
\begin{pmatrix}
  t \\
  f(t)
\end{pmatrix}
= \sum_{i=1}^{m} \begin{pmatrix}
  \xi^*_i \\
  c_i
\end{pmatrix} \varphi_i(t),
\]
\( (\xi^*_i, c_i)^T \in \mathbb{R}^2 \) control points
Notation: control polygon

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- curves in \( \mathbb{R}^d \)
  \[ C(t) := \sum_{i=1}^{m} c_i \varphi_i(t), \]
  \( c_i \in \mathbb{R}^d \) control points

- scalar functions
  \[ f(t) := \sum_{i=1}^{m} c_i \varphi_i(t), \]
  \[ \begin{pmatrix} t \\ f(t) \end{pmatrix} = \sum_{i=1}^{m} \begin{pmatrix} \xi_i^* \\ c_i \end{pmatrix} \varphi_i(t), \]
  \( (\xi_i^*, c_i)^T \in \mathbb{R}^2 \) control points
Bernstein polynomials

\[ B^{(p)}_i(t) := \binom{p-1}{i-1} t^{i-1} (1 - t)^{p-i} \]

\[ i = 1, \cdots, p, \, t \in [0, 1] \]
Bernstein polynomials

\[ B_i^{(p)}(t) := \binom{p - 1}{i - 1} t^{i-1} (1 - t)^{p-i} \]

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Bernstein polynomials $p : \text{order} = \text{degree} + 1$

$$B_i^{(p)}(t) := \binom{p - 1}{i - 1} t^{i-1} (1 - t)^{p-i}$$

$i = 1, \cdots, p, \ t \in [0, 1]$

- positivity: $B_i^{(p)}(t) \geq 0, \ t \in [0, 1]$
Bernstein polynomials $B_i^{(p)}(t) := \binom{p-1}{i-1} t^{i-1} (1 - t)^{p-i}$

- $i = 1, \cdots, p$, $t \in [0, 1]$

  - positivity: $B_i^{(p)}(t) \geq 0$, $t \in [0, 1]$

  - partition of unity: $\sum_{i=1}^{p} B_i^{(p)}(t) = 1$, $\forall t$
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**Computational consequences:** well conditioned basis
Bernstein polynomials $p: \text{order}=\text{degree}+1$

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- **partition of unity:** $\sum_{i=1}^{p} B_i^{(p)}(t) = 1$, $\forall t$

**Geometric consequences**

- p. unity $\Rightarrow$ affine invariance

$$p_i \in \mathbb{R}^d, \ C(t) = \sum_{i=1}^{p} p_i B_i^{(p)}(t) \Rightarrow AC(t) + q = \sum_{i=1}^{p} (Ap_i + q) B_i^{(p)}(t)$$
Bernstein polynomials \( p : \text{order} = \text{degree} + 1 \)

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B_i^{(p)}(t) := \binom{p-1}{i-1} t^{i-1} (1 - t)^{p-i}
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\]

- positivity & p. unity \( \Rightarrow \) convex hull

\[
C(t) = \sum_{i=1}^{p} p_{i-1} B_i^{(p)}(t) \in \mathbb{R}^d
\]
Bernstein polynomials: recurrence relations

\[ B_2^{(3)}(t) = 2t(1 - t) = (1 - t)t + t(1 - t) = (1 - t)B_2^{(2)}(t) + tB_1^{(2)}(t) \]
Bernstein polynomials: recurrence relations

\[ B_i^{(p)}(t) = (1 - t)B_i^{(p-1)}(t) + tB_{i-1}^{(p-1)}(t) \]

\[ B_i^{(k)} = 0, \quad i < 1, \quad i > k, \quad B_i^{(k)}(t) = B_i^{(k)}(t) := \binom{k - 1}{i - 1} t^{i-1}(1 - t)^{k-i}, \quad i = 1, \ldots, k \]
Bernstein polynomials: recurrence relations

\[ B_i^{(p)}(t) = (1 - t)B_i^{(p-1)}(t) + tB_{i-1}^{(p-1)}(t) \]

\[ \sum_{i=1}^{p} c_i^{(p)} B_i^{(p)}(t) = \sum_{i=1}^{p} c_i^{(p)} [(1-t)B_i^{(p-1)}(t)+tB_{i-1}^{(p)}(t)] = \sum_{i=1}^{p-1} c_i^{(p-1)} B_i^{(p-1)}(t) = \ldots = c_1^{(1)} \]

\[ c_{i}^{(k-1)} := (1 - t)c_i^{(k)} + tc_{i+1}^{(k)} \quad i = 1, \ldots, k - 1 \]
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de Casteljau algorithm

evaluation=convex combinations
Bernstein polynomials:

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Degree raising/Degree elevation

\[ \mathbb{P}_p \subset \mathbb{P}_{p+1} \Rightarrow \sum_{i=1}^{p} c_i B_i^{(p)}(t) = \sum_{i=1}^{p+1} \hat{c}_i B_i^{(p+1)}(t) \]
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- **Derivatives:**

\[ C(t) := \sum_{i=1}^{p} c_i B_i^{(p)}(t) \Rightarrow C'(t) = \sum_{i=1}^{p-1} (p - 1)(c_{i+1} - c_i)B_i^{(p-1)}(t) \]

\[ D(B_i^{(p)})(t) = (p - 1)[B_{i-1}^{(p-1)}(t) - B_i^{(p-1)}(t)] \]
Bernstein polynomials:

\[ B_i^{(p)}(t) = (1 - t)B_i^{(p-1)}(t) + tB_{i-1}^{(p-1)}(t) \]

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  \[ D(B_i^{(p)})(t) = (p-1)[B_{i-1}^{(p-1)}(t) - B_i^{(p-1)}(t)] \]

- **Integration:**

  \[ \int_0^1 B_i^{(p)}(t)dt = \frac{1}{p} \]

  \[ B_i^{(p)}(t) = \left[ \frac{\int_0^t B_{i-1}^{(p-1)}(s)ds}{\int_0^1 B_i^{(p-1)}(t)dt} - \frac{\int_0^t B_i^{(p-1)}(s)ds}{\int_0^1 B_i^{(p-1)}(t)dt} \right] \]
Bernstein polynomials ...

$B_1^{(p)} \cdots B_p^{(p)}$ is a Totally Positive (TP) basis for $\mathbb{P}_p$
TP bases

A basis \( \{ \varphi_1, \cdots, \varphi_p \} \) of a space \( \mathcal{U} \) is TP in \( I \subset \mathbb{R} \) if any collocation matrix

\[
\begin{pmatrix}
\varphi_1(t_1) & \cdots & \varphi_p(t_1) \\
\vdots & & \vdots \\
\varphi_1(t_r) & \cdots & \varphi_p(t_r)
\end{pmatrix}
\]

is Totally Positive (\( \Leftrightarrow \) any subdeterminant \( \geq 0 \))

\[ t_1 < t_2 < \cdots < t_r, \quad t_i \in I, \quad i = 1, \cdots, r \]
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\(t_1 < t_2 < \cdots < t_r, \ t_i \in I, \ i = 1, \cdots, r\)

is Totally Positive (\(\iff\) any subdeterminant \(\geq 0\))

A matrix is TP iff it is the product of positive “bidiagonal” matrices

\[
\downarrow
\]

\(\{\varphi_1, \cdots, \varphi_m\} \) TP basis \(c_1, \cdots, c_m \in \mathbb{R} \Rightarrow
\# \ \text{sign ch.} \ (\sum_{i=1}^{m} c_i \varphi_i(t)) \leq \# \ \text{sign ch.} \ (c_1, \cdots, c_m).\)
TP bases

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\( t_1 < t_2 < \cdots < t_r \), \( t_i \in I \), \( i = 1, \cdots, r \)

is Totally Positive (\( \iff \) any subdeterminant \( \geq 0 \))

TP \( \implies \) Variation diminishing

any hyperplane crosses

\[
\sum_{i=1}^{m} c_i \varphi_i(t)
\]

at most as many times as

its control polygon
Bernstein polynomials ...

- \( B_1^{(p)} \cdots B_p^{(p)} \) is a Totally Positive (TP) basis for \( \mathbb{P}_p \)
Bernstein polynomials ...

- $B_1^{(p)} \cdots B_p^{(p)}$ is a **Totally Positive** (TP) basis for $\mathbb{P}_p$

- $B_1^{(p)} \cdots B_p^{(p)}$ is the **best** normalized TP basis $\mathbb{P}_p$; it is the normalized $B$-basis for $\mathbb{P}_p$
Bernstein polynomials ...

- \( B_1^{(p)} \cdots B_p^{(p)} \) is a **Totally Positive** (TP) basis for \( \mathbb{IP}_p \)

- \( B_1^{(p)} \cdots B_p^{(p)} \) is the **best** normalized TP basis \( \mathbb{IP}_p \) it is the normalized B-basis for \( \mathbb{IP}_p \)

A (normalized and) TP basis \{\( B_1, \cdots, B_p \)\} of \( \mathcal{U} \) is a **B-basis** if for any (normalized and) TP basis \{\( \varphi_1, \cdots, \varphi_p \)\} of \( \mathcal{U} \) the matrix \( K \)

\[
\begin{pmatrix} 
\varphi_1 \\
\vdots \\
\varphi_p 
\end{pmatrix} = \begin{pmatrix} 
B_1 \\
\cdots \\
B_p 
\end{pmatrix} K
\]

is (stochastic and) TP.
Bernstein polynomials ...

- $B_1^{(p)} \cdots B_p^{(p)}$ is a Totally Positive (TP) basis for $\mathbb{P}_p$

- $B_1^{(p)} \cdots B_p^{(p)}$ is the best normalized TP basis $\mathbb{P}_p$; it is the normalized B-basis for $\mathbb{P}_p$

\[ \downarrow \]

- has minimum condition number (among p. bases)

$\Phi := (\varphi_1, \cdots, \varphi_p)$ positive basis of $\mathbb{P}_p$, $q(t) := \sum_{i=1}^{p} c_i \varphi_i(t)$

\[ C_\Omega(q(t), t) := \sum_{i=1}^{p} |c_i| \varphi_i(t) \quad \text{Condition number of } q \text{ w.r.t. } \Phi \]
Bernstein polynomials ...

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\[
\downarrow
\]

- has minimum condition number (among p. bases)

- has optimal geometric properties (among TP bases)
B-Bases: geometric properties

\{B_1^{(p)}, \ldots, B_p^{(p)}\} B-basis, \{\varphi_1, \ldots, \varphi_p\} TP basis

C(t) := \sum_{j=0}^{n} q_j \varphi_j(t) = \sum_{i=1}^{p} p_i B_i^{(p)}(t),

\downarrow

\begin{align*}
p_1, \ldots, p_p & \text{ “lies between” } q_1, \ldots, q_p \text{ and } C \\
\downarrow
\end{align*}
B-Bases: geometric properties

\[ \{ B_1^{(p)}, \ldots, B_p^{(p)} \} \text{ B-basis, } \{ \varphi_1, \ldots, \varphi_p \} \text{ TP basis} \]

\[ C(t) := \sum_{j=0}^{n} q_j \varphi_j(t) = \sum_{i=1}^{p} p_i B_i^{(p)}(t), \]

\[ \downarrow \]

\[ p_1, \ldots, p_p \text{ “lies between” } q_1, \ldots, q_p \text{ and } C \]

\[ \downarrow \]

\[ p_1, \ldots, p_p \text{ better description of } C \text{ : shape less “overemphasized”} \]
Bernstein polynomials ...

$B_{1}^{(p)} \cdots B_{p}^{(p)}$ is a Totally Positive (TP) basis for $\mathbb{IP}_{p}$
Bernstein polynomials ...

- \( B_1^{(p)} \cdots B_p^{(p)} \) is a Totally Positive (TP) basis for \( \mathbb{P}_p \)

- \( B_1^{(p)} \cdots B_p^{(p)} \) is the normalized B-basis for \( \mathbb{P}_p \)

\[
\inf \left\{ \frac{B_j^{(p)}(t)}{B_i^{(p)}(t)} : B_i^{(p)}(t) \neq 0 \right\} = 0, \ \forall \ i \neq j
\]

\[
\downarrow
\]

it has optimal geometric and computational properties
Bernstein polynomials ...

- $B_{1}^{(p)} \cdots B_{p}^{(p)}$ is a Totally Positive (TP) basis for $\mathbb{P}_{p}$

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\[ \inf \left\{ \frac{B_{i}^{(p)}(t)}{B_{i}^{(p)}(t)} : B_{i}^{(p)}(t) \neq 0 \right\} = 0, \forall i \neq j \]

\[ \Rightarrow \]

it has optimal geometric and computational properties

- all the given properties can be obtained by blossoms
B-splines

\[ \Xi := \{ \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p} \} \]

knots

\[ \cdots < \xi_i = \xi_{i+1} = \cdots = \xi_{i+p-1} < \xi_{i+p}, \quad 1 \leq \rho_i \leq p \]

multiplicity

usually

\[ \xi_1 = \cdots = \xi_p < \cdots < \xi_{n+1} = \cdots = \xi_{n+p} \]

\[ B_{i,\Xi}^{(1)}(t) := \begin{cases} 
1 & \text{if } t \in [\xi_i, \xi_{i+1}) \\
0 & \text{elsewhere} 
\end{cases} \]

\[ B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t) \]

\( i \)-th B-spline, of order \( p \), with knots \( \Xi \)
\[ \Xi := \{ \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p} \} \]

\[ \cdots < \xi_i = \xi_{i+1} = \cdots = \xi_{i+\rho_i - 1} < \xi_{i+\rho_i}, \quad 1 \leq \rho_i \leq p \]

usually

\[ \xi_1 = \cdots = \xi_p < \cdots < \xi_{n+1} = \cdots = \xi_{n+p} \]

\[
B^{(1)}_{i,\Xi}(t) := \begin{cases} 
1 & \text{if } t \in [\xi_i, \xi_{i+1}) \\
0 & \text{elsewhere}
\end{cases}
\]

\[
B^{(p)}_{i,\Xi}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B^{(p-1)}_{i,\Xi}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B^{(p-1)}_{i+1,\Xi}(t)
\]
B-splines

\[ \Xi := \left\{ \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p} \right\} \quad \text{knots} \]

\[ \cdots < \xi_i = \xi_{i+1} = \cdots = \xi_{i+\rho_i - 1} < \xi_{i+\rho_i}, \quad 1 \leq \rho_i \leq p \quad \text{multiplicity} \]

usually

\[ \xi_1 = \cdots = \xi_p < \cdots < \xi_{n+1} = \cdots = \xi_{n+p} \]

\[ B_{i,\Xi}^{(1)}(t) := \begin{cases} 1 & \text{if } t \in [\xi_i, \xi_{i+1}) \\ 0 & \text{elsewhere} \end{cases} \]

\[ B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t) \]
B-splines: evaluation

\[
B^{(p)}_{i,\Xi}(t) := \omega_{i,p}(t) B^{(p-1)}_{i,\Xi}(t) + (1 - \omega_{i+1,p}(t)) B^{(p-1)}_{i+1,\Xi}(t)
\]

\[
\omega_{i,p}(t) := \frac{t - \xi_i}{\xi_{i+p-1} - \xi_i}
\]

\[
B^{(p)}_{i,\Xi}(t) := \frac{t - \xi_i}{\xi_{i+p-1} - \xi_i} B^{(p-1)}_{i,\Xi}(t) + \frac{\xi_{i+p} - t}{\xi_{i+p} - \xi_{i+1}} B^{(p-1)}_{i+1,\Xi}(t)
\]

\[
\omega_{i+1,p}(t) := \frac{t - \xi_{i+1}}{\xi_{i+p} - \xi_{i+1}}
\]
B-splines: evaluation

\[
B_i^{(p)}(t) := \frac{t - \xi_i}{\xi_{i+p-1} - \xi_i} B_i^{(p-1)}(t) + \frac{\xi_{i+p} - t}{\xi_{i+p} - \xi_{i+1}} B_{i+1}^{(p-1)}(t)
\]

\[
B_i^{(p)}(t) := \omega_{i,p}(t) B_i^{(p-1)}(t) + (1 - \omega_{i+1,p}(t)) B_{i+1}^{(p-1)}(t)
\]

\[
C(t) := \sum c_i^{(p)} B_i^{(p)}(t), \quad t \in [\xi_r, \xi_{r+1})
\]
B-splines: evaluation

\[ B^{(p)}_{i,\Xi}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B^{(p-1)}_{i,\Xi}(t) + \frac{\xi_i+p-t}{\xi_{i+p-1}-\xi_{i+1}} B^{(p-1)}_{i+1,\Xi}(t) \]

\[ B^{(p)}_{i,\Xi}(t) := \omega_{i,p}(t) B^{(p-1)}_{i,\Xi}(t) + (1 - \omega_{i+1,p}(t)) B^{(p-1)}_{i+1,\Xi}(t) \]

\[ \omega_{i,p}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} \]

\[ C(t) := \sum_i c_i^{(p)} B^{(p)}_{i,\Xi}(t), \quad t \in [\xi_r, \xi_{r+1}) \]

\[ \sum_i c_i^{(p)} B^{(p)}_{i,\Xi}(t) = \sum_i c_i [\omega_{i,p}(t) B^{(p-1)}_{i,\Xi}(t) + (1-\omega_{i+1,p}(t)) B^{(p-1)}_{i+1,\Xi}(t)] = \sum_j c_j^{(p-1)} B^{(p-1)}_{j,\Xi}(t) \]

\[ = \cdots = \sum_j c_j^{(1)} B^{(1)}_{j,\Xi}(t) = c_r^{(1)} \]
B-splines: evaluation

\[
B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t)
\]

\[
B_{i,\Xi}^{(p)}(t) := \omega_{i,p}(t) B_{i,\Xi}^{(p-1)}(t) + (1 - \omega_{i+1,p}(t)) B_{i+1,\Xi}^{(p-1)}(t)
\]

\[
\omega_{i,p}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i}
\]

\[
C(t) := \sum \mathbf{c}_i^{(p)} B_{i,\Xi}^{(p)}(t), \quad t \in [\xi_r, \xi_{r+1})
\]

\[
\sum_i \mathbf{c}_i^{(p)} B_{i,\Xi}^{(p)}(t) = \sum_i \mathbf{c}_i [\omega_{i,p}(t) B_{i,\Xi}^{(p-1)}(t) + (1 - \omega_{i+1,p}(t)) B_{i+1,\Xi}^{(p-1)}(t)] = \sum_j \mathbf{c}_j^{(p-1)} B_{j,\Xi}^{(p-1)}(t)
\]

\[
= \cdots = \sum_j \mathbf{c}_j^{(1)} B_{j,\Xi}^{(1)}(t) = \mathbf{c}_r^{(1)}
\]

\[
\mathbf{c}_i^{(k-1)} := \omega_{i,k}(t) \mathbf{c}_i^{(k)} + (1 - \omega_{i,k}(t)) \mathbf{c}_{i-1}^{(k)}, \quad k = p, p-1, \ldots
\]
B-splines: evaluation

\[ B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t) \]

\[ B_i^{(p)}(t) := \omega_{i,p}(t) B_{i,\Xi}^{(p-1)}(t) + (1 - \omega_{i+1,p}(t)) B_{i+1,\Xi}^{(p-1)}(t) \]

\[ \omega_{i,p}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} \]

\[ C(t) := \sum_i c_i^{(p)} B_{i,\Xi}^{(p)}(t), \quad t \in [\xi_r, \xi_{r+1}) \]

\[ \sum_i c_i^{(p)} B_{i,\Xi}^{(p)}(t) = \sum_i c_i [\omega_{i,p}(t) B_{i,\Xi}^{(p-1)}(t) + (1-\omega_{i+1,p}(t)) B_{i+1,\Xi}^{(p-1)}(t)] = \sum_j c_j^{(p-1)} B_{j,\Xi}^{(p-1)}(t) \]

\[ = \cdots = \sum_j c_j^{(1)} B_{j,\Xi}^{(1)}(t) = c_r^{(1)} \]

\[ c_i^{(k-1)} := \omega_{i,k}(t)c_i^{(k)} + (1 - \omega_{i,k}(t))c_{i-1}^{(k)}, \quad k = p, p-1, \ldots \]

convex combination
B-splines: evaluation

$$B_{i,\Xi}^{(p)}(t) := \begin{cases} \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}^{}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t) & \text{if } i = 0 \\
(1 - \omega_{i+1,p}(t)) B_{i+1,\Xi}^{(p-1)}(t) & \text{if } i = n \\
\omega_{i,p}(t) B_{i,\Xi}^{(p-1)}(t) & \text{if } i = 0, 1, \ldots, n-1 \end{cases}$$

$$\omega_{i,p}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i}$$

$$C(t) := \sum c_i^{(p)} B_{i,\Xi}^{(p)}(t), \quad t \in [\xi_r, \xi_{r+1})$$

$$\sum_i c_i^{(p)} B_{i,\Xi}^{(p)}(t) = \sum_i c_i [\omega_{i,p}(t) B_{i,\Xi}^{(p-1)}(t) + (1 - \omega_{i+1,p}(t)) B_{i+1,\Xi}^{(p-1)}(t)] = \sum_j c_j^{(p-1)} B_{j,\Xi}^{(p-1)}(t)$$

$$= \cdots = \sum_j c_j^{(1)} B_{j,\Xi}^{(1)}(t) = c_r^{(1)}$$

$$c_i^{(k-1)} := \omega_{i,k}(t)c_i^{(k)} + (1 - \omega_{i,k}(t))c_i^{(k)}_{i-1}, \quad k = p, p-1, \ldots$$

convex combination

de Boor Algorithm
B-splines: properties

\[
B_i^{(p)}(t) := \frac{t - \xi_i}{\xi_{i+p-1} - \xi_i} B_i^{(p-1)}(t) + \frac{\xi_{i+p} - t}{\xi_{i+p} - \xi_{i+1}} B_{i+1}^{(p-1)}(t)
\]
**B-splines: properties**

\[ B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t) \]

- p.p. \( B_{i,\Xi}^{(p)} \in \mathbb{P}_p, \ t \in [\xi_j, \xi_{j+1}] \)
B-splines: properties

\[ B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t) \]

- p.p. \( B_{i,\Xi}^{(p)} \in \mathbb{P}_p, \ t \in [\xi_j, \xi_{j+1}] \)
- positivity \( B_{i,\Xi}^{(p)} \geq 0 \)
### B-splines: properties

\[
B_{i,\Xi}^{(p)}(t) := \frac{t - \xi_i}{\xi_{i+p-1} - \xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p} - t}{\xi_{i+p} - \xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t)
\]

- **p.p.** \( B_{i,\Xi}^{(p)} \in \mathbb{P}_p, \ t \in [\xi_j, \xi_{j+1}] \)
- **positivity** \( B_{i,\Xi}^{(p)} \geq 0 \)
- **compact (minimum) support**

\[
B_{i,\Xi}^{(p)}(t) = 0, \ t \notin [\xi_i, \xi_{i+p}]
\]

\[
B_{i,\Xi}^{(p)}(t) = 0, \ t \in [\xi_r, \xi_{r+1}], \ i \neq r, r-1, \ldots, r-p+1
\]
B-splines: a special case

\[ \Xi := \{ \xi_1 = 0 = \cdots = 0 = \xi_p < \xi_{p+1} = 1 = \cdots = 1 = \xi_{p+p} \} \]

knots

\[ B_{i,\Xi}^{(1)}(t) := \begin{cases} 
1 & \text{if } t \in [\xi_i, \xi_{i+1}) \\
0 & \text{elsewhere} 
\end{cases} \]

\[ B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t) \]
B-splines: a special case

\[ \Xi := \{ \xi_1 = 0 = \cdots = 0 = \xi_p < \xi_{p+1} = 1 = \cdots = 1 = \xi_{p+p} \} \]

knots

\[
B_{i,\Xi}^{(1)}(t) := \begin{cases} 
1 & \text{if } t \in [\xi_i, \xi_{i+1}) \\
0 & \text{elsewhere}
\end{cases}
\]

\[
B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t)
\]
B-splines: a special case

\[ \Xi := \{ \xi_1 = 0 = \cdots = 0 = \xi_p < \xi_{p+1} = 1 = \cdots = 1 = \xi_{p+p} \} \]

knots

\[ B_{i,\Xi}^{(1)}(t) := \begin{cases} 1 & \text{if } t \in [\xi_i, \xi_{i+1}) \\ 0 & \text{elsewhere} \end{cases} \]

\[ B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t) \]
B-splines: a special case

\[ \Xi := \{ \xi_1 = 0 = \cdots = 0 = \xi_p < \xi_{p+1} = 1 = \cdots = 1 = \xi_{p+p} \} \]

knots

\[ B_{i,\Xi}^{(1)}(t) := \begin{cases} 
1 & \text{if } t \in [\xi_i, \xi_{i+1}) \\
0 & \text{elsewhere} 
\end{cases} \]

\[ B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t) \]
B-splines: a special case

\[ \Xi := \{ \xi_1 = 0 = \cdots = 0 = \xi_p < \xi_{p+1} = 1 = \cdots = 1 = \xi_{p+p} \} \]  

knots

\[
B^{(1)}_{i, \Xi}(t) := \begin{cases} 
1 & \text{if } t \in [\xi_i, \xi_{i+1}) \\
0 & \text{elsewhere}
\end{cases}
\]

\[
B^{(p)}_{i, \Xi}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B^{(p-1)}_{i, \Xi}(t) + \frac{\xi_{i+p} - t}{\xi_{i+p} - \xi_{i+1}} B^{(p-1)}_{i+1, \Xi}(t)
\]

Bernstein polynomials
B-splines: properties ...

\[ B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t) \]
B-splines: properties...

\[ B_{i,\Xi}^{(p)}(t) := \omega_{i,p}(t)B_{i,\Xi}^{(p-1)}(t) + (1 - \omega_{i+1,p}(t))B_{i+1,\Xi}^{(p-1)}(t) \]

\[ \omega_{i,p}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} \]
B-splines: properties ...

\[ B_{i,\Xi}^{(p)}(t) := \omega_{i,p}(t) B_{i,\Xi}^{(p-1)}(t) + (1 - \omega_{i+1,p}(t)) B_{i+1,\Xi}^{(p-1)}(t) \]

\[ \omega_{i,p}(t) := \frac{t - \xi_i}{\xi_{i+p-1} - \xi_i} \]

\[ \psi_{ip}(\tau) := (\xi_{i+1} - \tau) \cdots (\xi_{i+p-1} - \tau), \quad \psi_{i1}(\tau) = 1 \]

\[ \omega_{ip}\psi_{ip}(\tau) + (1 - \omega_{ip})\psi_{i-1,p} = (t - \tau)\psi_{i,p-1}(\tau) \]
B-splines: properties ...

\[ B^{(p)}_{i,\Xi}(t) := \omega_{i,p}(t)B^{(p-1)}_{i,\Xi}(t) + (1 - \omega_{i+1,p}(t))B^{(p-1)}_{i+1,\Xi}(t) \]

\[ \omega_{i,p}(t) := \frac{t - \xi_i}{\xi_{i+p-1} - \xi_i} \]

\[ \psi_{ip}(\tau) := (\xi_{i+1} - \tau) \cdots (\xi_{i+p-1} - \tau), \quad \psi_{i1}(\tau) = 1 \]

\[ \omega_{ip}\psi_{ip}(\tau) + (1 - \omega_{ip})\psi_{i-1,p} = (t - \tau)\psi_{i,p-1}(\tau) \]

\[ \sum_i \psi_{ip}(\tau)B^{(p)}_{i,\Xi}(t) = \sum_i \psi_{ip}(\tau)[(\omega_{i,p}(t)B^{(p-1)}_{i,\Xi}(t) + (1 - \omega_{i+1,p}(t))B^{(p-1)}_{i+1,\Xi}(t)] = \]

\[ \sum_i [\omega_{ip}\psi_{ip}(\tau) + (1 - \omega_{ip})\psi_{i-1,p}]B^{(p-1)}_{i,\Xi}(t) = \]

\[ (t - \tau)\sum_{i=1}^n \psi_{i,p-1}(\tau)B^{(p-1)}_{i,\Xi}(t) = \ldots = (t - \tau)^{p-1}\sum_i \psi_{i,1}(\tau)B^{(1)}_{i,\Xi}(t) \]
B-splines: properties ...

\[ B_{i,\Xi}^{(p)}(t) := \omega_{i,p}(t) B_{i,\Xi}^{(p-1)}(t) + (1 - \omega_{i+1,p}(t)) B_{i+1,\Xi}^{(p-1)}(t) \]

\[ \omega_{i,p}(t) := \frac{t-\xi_i}{\xi_{i+p-1} - \xi_i} \]

\[ \psi_{ip}(\tau) := (\xi_{i+1} - \tau) \cdots (\xi_{i+p-1} - \tau), \quad \psi_{i1}(\tau) = 1 \]

\[ \omega_{ip}\psi_{ip}(\tau) + (1 - \omega_{ip})\psi_{i-1,p} = (t - \tau)\psi_{i,p-1}(\tau) \]

\[ \sum_i \psi_{ip}(\tau) B_{i,\Xi}^{(p)}(t) = \sum_i \psi_{ip}(\tau) [(\omega_{i,p}(t) B_{i,\Xi}^{(p-1)}(t) + (1 - \omega_{i+1,p}(t)) B_{i+1,\Xi}^{(p-1)}(t)] = \]

\[ \sum_i [\omega_{ip}\psi_{ip}(\tau) + (1 - \omega_{ip})\psi_{i-1,p}] B_{i,\Xi}^{(p-1)}(t) = \]

\[ (t - \tau) \sum_{i=1}^{n} \psi_{i,p-1}(\tau) B_{i,\Xi}^{(p-1)}(t) = \cdots = (t - \tau)^{p-1} \sum_i \psi_{i,1}(\tau) B_{i,\Xi}^{(1)}(t) \]

\[ (t - \tau)^{p-1} = \sum_i \psi_{ip}(\tau) B_{i,\Xi}^{(p)}(t), \quad t \in [\xi_p, \xi_{n+1}] \]
B-splines: properties ...

\[(t - \tau)^{p-1} = \sum_i \psi_{ip}(\tau) B_{i,\Xi}^{(p)}(t), \quad t \in [\xi_p, \xi_{n+1}]\]
B-splines: properties ...

\[(t - \tau)^{p-1} = \sum_i \psi_{ip}(\tau) B_{i;\Xi}^{(p)}(t), \quad t \in [\xi_p, \xi_{n+1}]\]

- Taylor exp. ⇒ Marsden’s identity:

\[q(t) = \sum_{i=1}^{n} \lambda_{ip}(q) B_{i;\Xi}^{(p)}(t), \quad \forall q \in \mathbb{P}_p, \ t \in [\xi_p, \xi_{n+1}]\]

\[\lambda_{ip}(f) := \sum_{r=1}^{p} \frac{(-D)^{r-1} \psi_{ip}(\tau)}{(p - 1)!} D^{p-r} f(\tau), \quad \psi_{ip}(\tau) := (\xi_{i+1} - \tau) \cdots (\xi_{i+p-1} - \tau)\]
B-splines: properties ...

\[(t - \tau)^{p-1} = \sum_i \psi_{ip}(\tau) B_{i, \Xi}^{(p)}(t), \ t \in [\xi_p, \xi_{n+1}]\]

- Taylor exp. \Rightarrow Marsden’s identity:

\[
q(t) = \sum_{i=1}^{n} \lambda_{ip}(q) B_{i, \Xi}^{(p)}(t), \ \forall q \in \mathbb{P}_p, \ t \in [\xi_p, \xi_{n+1}]
\]

\[
\lambda_{ip}(f) := \sum_{r=1}^{p} \frac{(-D)^{r-1} \psi_{ip}(\tau)}{(p-1)!} D^{p-r} f(\tau), \ \psi_{ip}(\tau) := (\xi_{i+1} - \tau) \cdots (\xi_{i+p-1} - \tau)
\]

- \(q(t) = 1 \Rightarrow \lambda_{ip}(q) = 1 \Rightarrow \) (local) p. unity

\[
1 = \sum_{i=1}^{n} B_{i, \Xi}^{(p)}(t), \ t \in [\xi_p, \xi_{n+1}],
\]

\[
1 = \sum_{r=i-p+1}^{i} B_{r, \Xi}^{(p)}(t), \ t \in [\xi_i, \xi_{i+1}]
\]
B-splines: properties ...

\[(t - \tau)^{p-1} = \sum_i \psi_{ip}(\tau) B_{i,\Xi}^{(p)}(t), \ t \in [\xi_p, \xi_{n+1}]\]

- Taylor exp. $\Rightarrow$ Marsden's identity:

\[
q(t) = \sum_{i=1}^{n} \lambda_{ip}(q) B_{i,\Xi}^{(p)}(t), \ \forall q \in \mathbb{P}_p, \ t \in [\xi_p, \xi_{n+1}]
\]

\[
\lambda_{ip}(f) := \sum_{r=1}^{p} \frac{(-D)^{r-1}\psi_{ip}(\tau)}{(p-1)!} D^{p-r} f(\tau), \ \psi_{ip}(\tau) := (\xi_{i+1} - \tau) \cdots (\xi_{i+p-1} - \tau)
\]

- $q(t) = t \Rightarrow \lambda_{ip}(q) = \frac{\xi_{i+1} + \cdots + \xi_{i+p-1}}{p-1} =: \xi_i^* \ Greville \ abscissas$

\[
t = \sum_{i=1}^{n} \xi_i^* B_{i,\Xi}^{(p)}(t), \ t \in [\xi_p, \xi_{n+1}]
\]
B-splines: properties ...

\[(t - \tau)^{p-1} = \sum_i \psi_{ip}(\tau) B_{i,\Xi}^{(p)}(t), \quad t \in [\xi_p, \xi_{n+1}]\]

- **Taylor exp.** ⇒ **Marsden’s identity:**

\[q(t) = \sum_{i=1}^{n} \lambda_{ip}(q) B_{i,\Xi}^{(p)}(t), \quad \forall q \in \mathbb{P}_p, \quad t \in [\xi_p, \xi_{n+1}]\]

\[\lambda_{ip}(f) := \sum_{r=1}^{p} \frac{(-D)^{r-1} \psi_{ip}(\tau)}{(p-1)!} D^{p-r} f(\tau), \quad \psi_{ip}(\tau) := (\xi_{i+1} - \tau) \cdots (\xi_{i+p-1} - \tau)\]

- **\(S_{\Xi,p} := \{\sum_j c_j B_{j,\Xi}^{(p)}\}**

\[f(t) = \sum_i \lambda_{ip}(f) B_{i,\Xi}^{(p)}(t), \quad \forall f \in S_{\Xi,p},\]

**projector:** de Boor-Fix
B-splines: properties ...

- positivity+ (local) p. unity ⇒ (local) convex hull
B-splines: properties ...

- positivity+ (local) p. unity ⇒ (local) convex hull
- curves in $\mathbb{R}^d$

\[ C(t) := \sum_{i=1}^{n} c_i B_{i,\Xi}^{(p)}(t), \quad c_i \in \mathbb{R}^d \text{ control point} \]

$t \in [\xi_i, \xi_{i+1}] \Rightarrow \text{only } c_{i-p+1}, \ldots c_{i-1}, c_i \text{ act}$
B-splines: properties ...

- positivity+ (local) p. unity $\Rightarrow$ (local) convex hull

- curves in $\mathbb{R}^d$

$$C(t) := \sum_{i=1}^{n} c_i B_i^{(p)}(t), \quad c_i \in \mathbb{R}^d \text{ control point}$$

$$t \in [\xi_i, \xi_{i+1}] \Rightarrow \text{only } c_{i-p+1}, \cdots, c_i \text{ active}$$

- scalar functions

$$\begin{pmatrix} t \\ f(t) \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix} \xi_i^* \\ c_i \end{pmatrix} B_i^{(p)}(t),$$

$$t \in [\xi_i, \xi_{i+1}] \Rightarrow \text{only } c_{i-p+1}, \cdots, c_{i-1}, c_i \text{ active}$$
B-splines: properties ...

- positivity+ (local) p. unity ⇒ (local) convex hull
  - curves in $\mathbb{R}^d$
    
    \[ C(t) := \sum_{i=1}^{n} c_i B_{i,\Xi}^{(p)}(t), \quad c_i \in \mathbb{R}^d \text{ control point} \]

    \[ t \in [\xi_i, \xi_{i+1}] \Rightarrow \text{only } c_{i-p+1}, \ldots, c_{i-1}, c_i \text{ active} \]

- scalar functions
  
  \[
  \begin{pmatrix}
  t \\
  f(t)
  \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix}
  \xi_i^* \\
  c_i
  \end{pmatrix} B_{i,\Xi}^{(p)}(t),
  \]

  \[ t \in [\xi_i, \xi_{i+1}] \Rightarrow \text{only } c_{i-p+1}, \ldots, c_{i-1}, c_i \text{ active} \]
B-splines: knots multiplicity

Given:

\[ \hat{\Xi} := \{\hat{\xi}_1 < \hat{\xi}_2 < \cdots < \hat{\xi}_{l+1}\}, \text{ break points} \]

\[ \nu := \{\nu_2, \cdots, \nu_l, \}, \text{ #smoothness – conditions} \]

\[ \mathcal{P}^{\nu}_{p,\hat{\Xi}} := \{ s : s|_{[\hat{\xi}_i, \hat{\xi}_{i+1}]} \in \mathbb{P}_p, s^{(j)}(\hat{\xi}_i^+) = s^{(j)}(\hat{\xi}_i^-), j = 0, \cdots, \nu_i - 1, i = 2, \cdots, l \} =: \text{p.p.} \]
B-splines: knots multiplicity

Given:

\[ \hat{\Xi} := \{ \hat{\xi}_1 < \hat{\xi}_2 < \cdots < \hat{\xi}_{l+1} \}, \text{break points} \]

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\[ \mathcal{P}_{p,\hat{\Xi}}^{\nu} := \{ s : s|_{[\hat{\xi}_i, \hat{\xi}_{i+1}]} \in \mathbb{P}_p, s^{(j)}(\hat{\xi}_i^+) = s^{(j)}(\hat{\xi}_i^-), j = 0, \cdots, \nu_i-1, i = 2, \cdots, l \} =: \text{p.p.} \]

Put:

\[ \Xi := \{ \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p} \} \]

- \[ \xi_1 \cdots < \xi_p \leq \hat{\xi}_1 \leq \cdots \leq \hat{\xi}_{l+1} \leq \xi_{n+1} \leq \xi_{n+p} \]

- \[ \hat{\xi}_j \text{, occurs } p - \nu_j \text{ times in } \Xi, \; j = 2, \cdots, l \]
B-splines: knots multiplicity

Given:

\[ \hat{\Xi} := \{\hat{\xi}_1 < \hat{\xi}_2 < \cdots < \hat{\xi}_{l+1}\}, \text{ break points} \]

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\[ \mathcal{P}^{\nu}_{p,\hat{\Xi}} := \{s : s|_{[\hat{\xi}_i, \hat{\xi}_{i+1}]} \in \mathbb{P}_p, s^{(j)}(\hat{\xi}^+_i) = s^{(j)}(\hat{\xi}^-_i), j = 0, \cdots, \nu_i - 1, i = 2, \cdots, l\} =: \text{p.p.} \]

Put:

\[ \Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p}\} \]

- \(\xi_1 \cdots < \xi_p \leq \hat{\xi}_1 \leq \cdots \leq \hat{\xi}_{l+1} \leq \xi_{n+1} \leq \xi_{n+p}\)
- \(\hat{\xi}_j\), occurs \(p - \nu_j\) times in \(\Xi\), \(j = 2, \cdots, l\)

\[ \Rightarrow \{B_1^{(p)}_{1,\Xi}, \cdots, B_n^{(p)}_{n,\Xi}\} \text{ is a basis of } \mathcal{P}^{\nu}_{p,\hat{\Xi}} \] (Curry-Schoenberg)
**B-splines: knots multiplicity**

**Given:**

\[ \hat{\Xi} := \{\hat{\xi}_1 < \hat{\xi}_2 < \cdots < \hat{\xi}_{l+1}\}, \text{ break points} \]

\[ \nu := \{\nu_2, \cdots, \nu_l\}, \# \text{ smoothness conditions} \]

\[ \mathcal{P}_p,\hat{\Xi}^\nu := \{s : s|_{[\hat{\xi}_i, \hat{\xi}_{i+1}]} \in \mathbb{P}_p, s^{(j)}(\hat{\xi}_i^+) = s^{(j)}(\hat{\xi}_i^-), j = 0, \cdots, \nu_i-1, i = 2, \cdots, l\} =: \text{p.p.} \]

**Put:**

\[ \Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p}\} \]

- \[ \xi_1 \cdots < \xi_p \leq \hat{\xi}_1 \leq \cdots \leq \hat{\xi}_{l+1} \leq \xi_{n+1} \leq \xi_{n+p} \]
- \[ \hat{\xi}_j, \text{ occurs } p - \nu_j \text{ times in } \Xi, \ j = 2, \cdots, l \]

**B-splines are a basis for p.p.**
B-splines: knots multiplicity

Given:

\[ \hat{\Xi} := \{\hat{\xi}_1 < \hat{\xi}_2 < \cdots < \hat{\xi}_{l+1}\}, \text{ break points} \]

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\[ \mathcal{P}_{\nu,\hat{\Xi}} := \{s : s|_{[\hat{\xi}_i, \hat{\xi}_{i+1}]} \in \mathbb{P}_{p}, s^{(j)}(\hat{\xi}_i^+) = s^{(j)}(\hat{\xi}_i^-), j = 0, \cdots, \nu_i - 1, i = 2, \cdots, l\} =: \text{p.p.} \]

Put:

\[ \Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p}\} \]

\[ \xi_1 \cdots < \xi_p \leq \hat{\xi}_1 \leq \cdots \leq \hat{\xi}_{l+1} \leq \xi_{n+1} \leq \xi_{n+p} \]

\[ \hat{\xi}_j, \text{ occurs } p - \nu_j \text{ times in } \Xi, \ j = 2, \cdots, l \]

B-splines are a basis for p.p.

\[ < B^{(p)}_{1,\Xi}, \cdots, B^{(p)}_{n,\Xi} > |_{[\xi_r,\xi_{r+1}]} = < B^{(p)}_{r-p+1,\Xi}, \cdots, B^{(p)}_{r,\Xi} > \equiv \mathbb{P}_p|_{[\xi_r,\xi_{r+1}]} \Rightarrow \]

\[ \{B^{(p)}_{1,\Xi}, \cdots, B^{(p)}_{n,\Xi}\} \text{ are locally lin. independent} \]
⋯ < \xi_j = \xi_{j+1} = \cdots = \xi_{j+\rho_i-1} < \xi_{j+\rho_j}, \ 1 \leq \rho_j \leq p \text{ multiplicity}

\begin{align*}
B_{i,\Xi}^{(p)} \text{ is (at least) of class } C^{p-\rho_j-1} \text{ at } \xi_j
\end{align*}

(p - \rho_j - 1 + 1) + \rho_j = p
\[
\cdots \xi_j = \xi_{j+1} = \cdots = \xi_j + \rho_i - 1 < \xi_j + \rho_j, \quad 1 \leq \rho_j \leq p \quad \text{multiplicity}
\]

\[
B_{i,\Xi}^{(p)} \quad \text{is (at least) of class } C^{p-\rho_j-1} \quad \text{at } \xi_j
\]

\[
(p - \rho_j - 1 + 1) + \rho_j = p
\]

\# smoothness cond. + knot multiplicity=order
B-splines: knots multiplicity ...

\[ \cdots < \xi_j = \xi_{j+1} = \cdots = \xi_{j+p_i-1} < \xi_{j+p}, \ 1 \leq \rho_j \leq p \ \text{multiplicity} \]

\[ B_{i,\Xi}^{(p)} \text{ is (at least) of class } C^{p-\rho_j-1} \text{ at } \xi_j \]

\[ (p - \rho_j - 1 + 1) + \rho_j = p \]

# smoothness cond. + knot multiplicity=order

\[ \Xi = [0, 1, 2, 3, 4, 5, 6, 7], [0, 1, 2, 3, 4, 5, 6, 7], [0, 1, 2, 4, 4, 5, 6, 7], [0, 1, 2, 4, 4, 6, 7] \]
B-splines: knot insertion

\[ \Xi := \{ \cdots \leq \xi_i \leq \xi_{i+1} \leq \cdots \} \subset \{ \cdots \leq \xi_i \leq \bar{\xi} \leq \xi_{i+1} \leq \cdots \} =: \bar{\Xi}, \quad S_{\Xi,p} := \left\{ \sum_j c_j B_j^{(p)} \right\} \]
B-splines: knot insertion

$$\Xi := \{ \cdots \leq \xi_i \leq \xi_{i+1} \leq \cdots \} \subset \{ \cdots \leq \xi_i \leq \bar{\xi} \leq \xi_{i+1} \leq \cdots \} =: \bar{\Xi}, \quad S_{\Xi,p} := \sum_j c_j B_j^{(p)}$$

$$S_{\Xi,p} \subset S_{\bar{\Xi},p}$$
B-splines: knot insertion

\[ \Xi := \{ \cdots \leq \xi_i \leq \xi_{i+1} \leq \cdots \} \subset \{ \cdots \leq \xi_i \leq \bar{\xi} \leq \xi_{i+1} \leq \cdots \} =: \bar{\Xi}, \quad S_{\Xi;p} := \{ \sum_j c_j B_j^{(p)}_{\Xi} \} \]

\[ S_{\Xi;p} \subset S_{\bar{\Xi},p} \]

\[ \sum_j c_j B_j^{(p)}_{\Xi} = \sum_j \bar{c}_j B_j^{(p)}_{\bar{\Xi}} \]

\[ \bar{c}_j = \gamma_{j,p} c_j + (1 - \gamma_{j,p}) c_{j-1} \]

\[ \gamma_{j,p} := \begin{cases} 
1 & \xi_{j+p-1} \leq \bar{\xi} \\
\frac{\bar{\xi} - \xi_j}{\xi_{j+p-1} - \xi_j} & \xi_j < \bar{\xi} < \xi_{j+p-1} \\
0 & \bar{\xi} \leq t_j 
\end{cases} \]
B-splines: knot insertion

$$\Xi := \{ \cdots \leq \xi_i \leq \xi_{i+1} \leq \cdots \} \subset \{ \cdots \leq \xi_i \leq \bar{\xi} \leq \xi_{i+1} \leq \cdots \} =: \bar{\Xi}, \quad S_{\Xi,p} := \{ \sum_j c_j B_{j,\Xi}^{(p)} \}$$

$$S_{\Xi,p} \subset S_{\bar{\Xi},p}$$

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1 & \xi_{j+p-1} \leq \bar{\xi} \\
\frac{\bar{\xi} - \xi_j}{\xi_{j+p-1} - \xi_j} & \xi_j < \bar{\xi} < \xi_{j+p-1} \\
0 & \bar{\xi} \leq t_j 
\end{cases}$$

the new c.p. interpolates the old one at the new Greville abscissas
the new contr. pol. interpolates the old one at the new Greville abscissas
B-splines: knot insertion ...

the new contr. pol. interpolates the old one at the new Greville abscissas
B-splines: knot insertion ...

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B-splines: knot insertion ...

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B-splines: iterating knot insertion
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B-splines: iterating knot insertion

The closer is the control polygon to the function being represented.
B-splines: iterating knot insertion

The knot sequence, the closer is the control polygon to the function being represented.
B-splines: iterating knot insertion

Corner cutting
the finer the knot sequence, the closer is the control polygon to the function being represented
B-splines: iterating knot insertion

the finer the knot sequence, the closer is the control polygon to the function being represented

\[ \| f - C_{p,\Xi} \| \leq \text{const} |\Xi|^2 \| D^2 f \|, \]

\[ |\Xi| := \max_i (\xi_{i+1} - \xi_i) \]
B-splines: iterating knot insertion

the finer the knot sequence, the closer is the control polygon to the function being represented

\[ \| f - C_{p,\Xi} \| \leq \text{const} |\Xi|^2 \| D^2 f \|, \]

\[ |\Xi| := \max_i (\xi_{i+1} - \xi_i) \]

\[ \Downarrow \]

convergence under knot refinement
B-splines: iterating knot insertion

the finer the knot sequence, the closer is the control polygon to the function being represented

\[ \| f - C_{p,\Xi} \| \leq \text{const} |\Xi|^2 \| D^2 f \|, \]

\[ |\Xi| := \max_i (\xi_{i+1} - \xi_i) \]

CAGD: knot insertion/knot refinement

↓

Isogeometric Analysis: h-refinement
the finer the knot sequence, the closer is the control polygon to the function being represented

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\[ |\Xi| := \max_i (\xi_{i+1} - \xi_i) \]

CAGD: knot insertion/knot refinement

\[ \uparrow \]

Isogeometric Analysis: \( h \)-refinement

knot refinement: successive convex combinations
B-splines: more on knot insertion

\[ \Xi := \{ \cdots \leq \xi_i \leq \xi_{i+1} \leq \cdots \} \subset \{ \cdots \leq \xi_i \leq \bar{\xi} \leq \xi_{i+1} \leq \cdots \} =: \bar{\Xi}, \quad S_{\Xi,p} := \sum_j c_j B_{j,\Xi}^{(p)} \]
B-splines: more on knot insertion

\[ \Xi := \{ \cdots \leq \xi_i \leq \xi_{i+1} \leq \cdots \} \subset \{ \cdots \leq \xi_i \leq \bar{\xi} \leq \xi_{i+1} \leq \cdots \} =: \bar{\Xi}, \quad S_{\Xi, p} := \{ \sum_j c_j B_{j, \Xi}^{(p)} \} \]

\[ \sum_j c_j B_{j, \Xi}^{(p)} = \sum_j \bar{c}_j B_{j, \Xi}^{(p)}, \quad \bar{c}_j = \gamma_{j,p} c_j + (1 - \gamma_{j,p}) c_{j-1}, \]

\[ \Downarrow \]

\[ \# \text{ sign ch. } (\cdots \bar{c}_j \cdots) \leq \# \text{ sign ch. } (\cdots, c_j, \cdots). \]
B-splines: more on knot insertion

\[ \Xi := \{ \cdots \leq \xi_i \leq \xi_{i+1} \leq \cdots \} \subset \{ \cdots \leq \xi_i \leq \bar{\xi} \leq \xi_{i+1} \leq \cdots \} =: \bar{\Xi}, \ S_{\Xi,p} := \{ \sum_j c_j B_{j,\Xi}^{(p)} \} \]

\[ \sum_j c_j B_{j,\Xi}^{(p)} = \sum_j \bar{c}_j B_{j,\bar{\Xi}}^{(p)}, \quad \bar{c}_j = \gamma_{j,p} c_j + (1 - \gamma_{j,p}) c_{j-1}, \]

\[ \exists \text{ sign ch. } (\cdots \bar{c}_j \cdots) \leq \# \text{ sign ch. } (\cdots, c_j, \cdots). \]

\[ x := \xi_i = \xi_{i+1} = \cdots = \xi_{i+p-2} < \xi_{i+p-1} \]

\[ \sum_j c_j B_{j,\Xi}^{(p)}(x) = c_{i-1} B_{i-1,\Xi}^{(p)}(x) = c_{i-1} \]
B-splines: more on knot insertion

\[ \Xi := \{ \cdots \leq \xi_i \leq \xi_{i+1} \leq \cdots \} \subset \{ \cdots \leq \xi_i \leq \bar{\xi} \leq \xi_{i+1} \leq \cdots \} =: \bar{\Xi}, \quad S_{\Xi, p} := \left\{ \sum_j c_j B_{j,\Xi}^{(p)} \right\} \]

\[ \sum_j c_j B_{j,\Xi}^{(p)} = \sum_j \tilde{c}_j B_{j,\Xi}^{(p)}, \quad \tilde{c}_j = \gamma_{j, p} c_j + (1 - \gamma_{j, p}) c_{j-1}, \]

\[ \# \text{ sign ch. } (\ldots \tilde{c}_j \ldots) \leq \# \text{ sign ch. } (\ldots, c_j, \ldots). \]

\[ x := \xi_i = \xi_{i+1} = \cdots = \xi_{i+p-2} < \xi_{i+p-1} \]

\[ \sum_j c_j B_{j,\Xi}^{(p)}(x) = c_{i-1} B_{i-1,\Xi}^{(p)}(x) = c_{i-1} \]

 evaluation at \( t = x \):
insert \( x \) in \( \Xi \) \( p - 1 \) times and take the coeff.
B-splines: more on knot insertion

\[ \Xi := \{ \cdots \leq \xi_i \leq \xi_{i+1} \leq \cdots \} \subset \{ \cdots \leq \xi_i \leq \bar{\xi} \leq \xi_{i+1} \leq \cdots \} =: \bar{\Xi}, \quad S_{\Xi, p} := \left\{ \sum_{j} c_j B_{j, \Xi}^{(p)} \right\} \]

\[ \sum_{j} c_j B_{j, \Xi}^{(p)} = \sum_{j} \bar{c}_j B_{j, \bar{\Xi}}^{(p)}, \quad \bar{c}_j = \gamma_{j, p} c_j + (1 - \gamma_{j, p}) c_{j-1}, \]

\[ \# \text{ sign ch. } (\cdots \bar{c}_j \cdots) \leq \# \text{ sign ch. } (\cdots, c_j, \cdots). \]

\[ x := \xi_i = \xi_{i+1} = \cdots = \xi_{i+p-2} < \xi_{i+p-1} \]

\[ \sum_{j} c_j B_{j, \Xi}^{(p)}(x) = c_{i-1} B_{i-1, \Xi}^{(p)}(x) = c_{i-1} \]

evaluation at \( t = x \):
insert \( x \) in \( \Xi \)
\( p - 1 \) times and take the coeff.

\[ \# \text{ sign ch. } (\sum_{j} c_j B_{j, \Xi}^{(p)}(t)) \leq \# \text{ sign ch. } (\cdots, c_j, \cdots) \]
B-splines: B-basis

\[ B_{1,\Xi}^{(p)} \cdots B_{n,\Xi}^{(p)} \] is a Totally Positive basis for \( S_{\Xi,p} \)
B-splines: B-basis

- $B_{1,\Xi}^{(p)} \cdots B_{n,\Xi}^{(p)}$ is a Totally Positive basis for $S_{\Xi,p}$
- $B_{1,\Xi}^{(p)} \cdots B_{n,\Xi}^{(p)}$ is the normalized B-basis for $S_{\Xi,p}$
**B-splines: B-basis**

- $B_{1,\Xi}^{(p)} \cdots B_{n,\Xi}^{(p)}$ is a **Totally Positive** basis for $S_{\Xi,p}$

- $B_{1,\Xi}^{(p)} \cdots B_{n,\Xi}^{(p)}$ is the **normalized B-basis** for $S_{\Xi,p}$

\[ \downarrow \]

- any collocation matrix is TP
- optimal **geometric and computational** properties
**B-splines: B-basis**

- \( B_{1,\Xi}^{(p)} \cdots B_{n,\Xi}^{(p)} \) is a **Totally Positive** basis for \( S_{\Xi,p} \)

- \( B_{1,\Xi}^{(p)} \cdots B_{n,\Xi}^{(p)} \) is the **normalized B-basis** for \( S_{\Xi,p} \)

\[ \downarrow \]

- any collocation matrix is TP
- optimal **geometric** and **computational** properties

- all the given properties can be obtained by blossoms
B-splines: degree elevation

\[ \mathcal{P}_p \subset \mathcal{P}_{p+1} \]
B-splines: degree elevation

\[ \mathbb{P}_p \subset \mathbb{P}_{p+1} \]

# smoothness cond. + knot multiplicity=order

\[ \Xi := \{ \cdots < \xi_r = \cdots = \xi_{r+\rho_r-1} < \cdots \} \subset \{ \cdots < \xi_r = \cdots = \xi_{r+\rho_r} < \cdots \} =: \bar{\Xi} \]
B-splines: degree elevation

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\# smoothness cond. + knot multiplicity=order

\[ \Xi := \{ \cdots < \xi_r = \cdots = \xi_{r+\rho_r-1} < \cdots \} \subset \{ \cdots < \xi_r = \cdots = \xi_{r+\rho_r} < \cdots \} =: \Xi \]

\[ S_{\Xi,p} \subset S_{\Xi,p+1} \]

\[ B_{i,[\xi_i,\ldots,\xi_{i+p}]}^{(p)} = \frac{1}{p} \sum_{r=i}^{i+p} B_{r,[\xi_i,\ldots,\xi_r,\xi_r,\ldots,\xi_{i+p}]}^{(p+1)} \]
B-splines: degree elevation

\[ \mathbb{P}_p \subset \mathbb{P}_{p+1} \]

# smoothness cond. + knot multiplicity=order

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\[ S_{\Xi, p} \subset S_{\Xi, p+1} \]

\[ B_{i,[\xi_i,\cdots,\xi_{i+p}]}^{(p)} = \frac{1}{p} \sum_{r=i}^{i+p} B_{r,[\xi_i,\cdots,\xi_r,\xi_r,\cdots,\xi_{i+p}]}^{(p+1)} \]
B-splines: iterating degree elevation

Repeated degree elevation leads to a sequence of control polynomials that converges to the spline

\[ \| f - C_{m,\Xi} \| = O \left( \frac{1}{m} \right), \quad f \in S_{\Xi,p}, \quad m > p \]
B-splines: iterating degree elevation

Repeated degree elevation leads to a sequence of control polynomials that converges to the spline.

\[ \| f - C_{m,\Xi} \| = O \left( \frac{1}{m} \right), \quad f \in \mathcal{S}_{\Xi,p}, \quad m > p \]

\[ \downarrow \]

Convergence under degree elevation.
B-splines: iterating degree elevation

Repeated degree elevation leads to a sequence of control. pol. that converges to the spline

$$\| f - C_{m,\Xi} \| = O \left( \frac{1}{m} \right), \quad f \in S_{\Xi,p}, \quad m > p$$

CAGD: degree elevation/degree raising

↓

Isogeometric Analysis: $p$-refinement
B-splines: iterating degree elevation

Repeated degree elevation leads to a sequence of control. pol. that converges to the spline

$$\|f - C_{m,\Xi}\| = O\left(\frac{1}{m}\right), \quad f \in S_{\Xi,p}, \quad m > p$$

CAGD: degree elevation/degree raising

↓

Isogeometric Analysis: $p$-refinement

degree elevation: blossoms
B-splines

\[ B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t) \]

**Derivatives:**

\[
D(\sum_i c_i B_{i,\Xi}^{(p)}(t)) = \sum_i (p - 1) \frac{c_i - c_{i-1}}{\xi_{i+p-1} - \xi_i} B_{i,\Xi}^{(p-1)}(t)
\]

\[
(B_{i,\Xi}^{(p)})'(t) = (p - 1)\left[\frac{1}{\xi_{i+p-1} - \xi_i} B_{i,\Xi}^{(p-1)}(t) - \frac{1}{\xi_{i+p} - \xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t)\right]
\]
B-splines

\[
B^{(p)}_{i,\Xi}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B^{(p-1)}_{i,\Xi}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B^{(p-1)}_{i+1,\Xi}(t)
\]

Derivatives:

\[
D(\sum_i c_i B^{(p)}_{i,\Xi}(t)) = \sum_i (p-1) \frac{c_i-c_{i-1}}{\xi_{i+p-1}-\xi_i} B^{(p-1)}_{i,\Xi}(t)
\]

\[
(B^{(p)}_{i,\Xi})'(t) = (p-1)[\frac{1}{\xi_{i+p-1}-\xi_i} B^{(p-1)}_{i,\Xi}(t) - \frac{1}{\xi_{i+p}-\xi_{i+1}} B^{(p-1)}_{i+1,\Xi}(t)]
\]

Integral recurrence relation:

\[
\int_{s} B^{(p)}_{i,\Xi}(t) \, ds = [B^{(p)}_{i,\Xi}(t)]^1_{-1} + \int_{s} B^{(p)}_{i+1,\Xi}(t) \, ds
\]

---

**Linear** **Quadratic** **Cubic**

\[
\begin{align*}
B_1(t) & = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
DB_1(t) & = \begin{cases} 1, & 0 \leq t < 1 \\ -1, & 1 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases} \\
D^2B_1(t) & = \begin{cases} 1, & 0 \leq t < 1 \\ -2, & 1 \leq t < 2 \\ 1, & 2 \leq t \leq 3 \\ 0, & \text{otherwise} \end{cases}
\end{align*}
\]

---

**2-fold** **3-fold** **4-fold**

\[
\begin{align*}
B_2(t) & = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
DB_2(t) & = \begin{cases} 1, & 0 \leq t < 1 \\ -1, & 1 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases} \\
D^2B_2(t) & = \begin{cases} 1, & 0 \leq t < 1 \\ -2, & 1 \leq t < 2 \\ 1, & 2 \leq t \leq 3 \\ 0, & \text{otherwise} \end{cases}
\end{align*}
\]
B-splines

\[ B_{i, \Xi}^{(p)}(t) := \frac{t - \xi_i}{\xi_{i+p-1} - \xi_i} B_{i, \Xi}^{(p-1)}(t) + \frac{\xi_{i+p} - t}{\xi_{i+p} - \xi_{i+1}} B_{i+1, \Xi}^{(p-1)}(t) \]

**Derivatives:**

\[ D(\sum_i c_i B_{i, \Xi}^{(p)}(t)) = \sum_i (p - 1) \frac{c_i - c_{i-1}}{\xi_{i+p-1} - \xi_i} B_{i, \Xi}^{(p-1)}(t) \]

\[ (B_{i, \Xi}^{(p)})'(t) = (p - 1) \left[ \frac{1}{\xi_{i+p-1} - \xi_i} B_{i, \Xi}^{(p-1)}(t) - \frac{1}{\xi_{i+p} - \xi_{i+1}} B_{i+1, \Xi}^{(p-1)}(t) \right] \]

**Integral recurrence relation:**

\[ B_i^{(p)}(t) = \int_{-\infty}^{t} \delta_{i, \Xi}^{(p-1)} B_{i, \Xi}^{(p-1)}(s) ds - \int_{-\infty}^{t} \delta_{i+1, \Xi}^{(p-1)} B_{i+1, \Xi}^{(p-1)}(s) ds \]

\[ \delta_{i, \Xi}^{(p)} := \frac{1}{\int_{-\infty}^{+\infty} B_{i, \Xi}^{(p)}(s) ds} \]
B-splines

\[ B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t) \]

Derivatives:

\[ D(\sum_i c_i B_{i,\Xi}^{(p+1)}(t)) = \sum_i p \frac{c_i - c_{i-1}}{\xi_{i+p} - \xi_i} B_{i,\Xi}^{(p)}(t) \]
B-splines

\[ B_{i,\Xi}^{(p)}(t) := \frac{t-\xi_i}{\xi_{i+p-1}-\xi_i} B_{i,\Xi}^{(p-1)}(t) + \frac{\xi_{i+p}-t}{\xi_{i+p}-\xi_{i+1}} B_{i+1,\Xi}^{(p-1)}(t) \]

- **Derivatives:**

\[ D \left( \sum_i c_i B_{i,\Xi}^{(p+1)}(t) \right) = \sum_i p \frac{c_i - c_{i-1}}{\xi_{i+p} - \xi_i} B_{i,\Xi}^{(p)}(t) \]

- **Integration:**

\[ \int_{\xi_1}^{t} \sum_{i=1}^{n} c_i B_{i,\Xi}^{(p)}(s) \, ds = \sum_{i=1}^{r-1} \left( \sum_{j=1}^{i} c_j \frac{\xi_{p+j} - \xi_j}{p} \right) B_{i,\Xi}^{(p+1)}(t) \]

with \( \xi_1 \leq t \leq \xi_r \)
Tensor-product B-splines surfaces

Ξ := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p}\}, \ \Upsilon := \{v_1 \leq v_2 \leq \cdots \leq v_{m+q}\}

\[ S(u, v) = \sum_{i=1, j=1}^{n, m} c_{i,j} B_{i,\Xi}^{(p)}(u) B_{i,\Upsilon}^{(q)}(v) \]
Tensor-product B-splines surfaces

\[ \Xi := \{ \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p} \}, \quad \Upsilon := \{ \nu_1 \leq \nu_2 \leq \cdots \leq \nu_{m+q} \} \]

\[ S(u, v) = \sum_{i=1, j=1}^{n,m} c_{i,j} B_{i,\Xi}^{(p)}(u) B_{i,\Upsilon}^{(q)}(v) \]
\[ \Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p}\}, \ \Upsilon := \{\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{m+q}\} \]

\[ S(u, v) = \sum_{i=1, j=1}^{n,m} c_{i,j} B_{i,\Xi}^{(p)}(u) B_{i,\Upsilon}^{(q)}(v) \]
Given \( \{ B_{i,\Xi}^{(p)}(t), \ i = 1, \cdots \} \), \( W := \{ w_i \geq 0, \ i = 1, \cdots \} \), weights \( R_{i,\Xi,W}^{(p)}(t) := \frac{w_i B_{i,\Xi}^{(p)}(t)}{\sum_{j=1}^{n} w_j B_{i,\Xi}^{(p)}(t)} \)

NURBS: projective transformation of B-splines
NURBS: projective transformation of B-splines

- positivity
- p. of unity
- compact support
- smoothness
- ...

Given $\{B_{i,\Xi}^{(p)}(t), i = 1, \cdots \}$, $W := \{w_i \geq 0, i = 1, \cdots \}$, weights

$$R_{i,\Xi, W}^{(p)}(t) := \frac{w_i B_{i,\Xi}^{(p)}(t)}{\sum_{j=1}^{n} w_i B_{j,\Xi}^{(p)}(t)}$$
NURBS: ...

- **quadratic NURBS** exactly represent (segments of) **conic sections**
  - Ex. unit circle \( \left( \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \)
NURBS: ...

- Quadratic NURBS exactly represent (segments of) conic sections
- Ex. unit circle \( \left( \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \)

- NURBS possess nice properties of B-splines and allow exact description of the geometry for a large set of objects of practical interest
NURBS: ...

- quadratic NURBS exactly represent (segments of) conic sections
- Ex. unit circle \( \left( \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \)

- NURBS possess nice properties of B-splines and allow exact description of the geometry for a large set of objects of practical interest

- NURBS are an efficient tool for Isogeometric Analysis
NURBS: parametrization of conic sections...

\[
\begin{pmatrix}
\frac{1-t^2}{1+t^2}, \\
\frac{2t}{1+t^2}
\end{pmatrix} : 200 \text{ “equispaced” points}
\]
NURBS: parametrization of conic sections ...

\[
\left( \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) : 200 \text{ “equispaced” points}
\]

\[
(\sin(t), \cos(t)) : 200 \text{ equispaced points}
\]
NURBS: parametrization of conic sections ...

NURBS parametrization $t \in [0, 1]:$

100 “equispaced” points
NURBS: parametrization of conic sections ...

NURBS parametrization $t \in [0, 1]$:
100 “equispaced” points

$(\sin(t), \cos(t)) : 100$ equispaced points
NURBS: parametrization of conic sections ...

NURBS parametrization \( t \in [0, 1] \):
100 “equispaced” points

\((\sin(t), \cos(t))\) : 100 equispaced points

only straight lines have rational repres. w.r.t. arc length
Drawbacks of the rational model

- Rational curves require additional parameters (weights) whose selection is often not clear.
- The derivative of a degree-$p$ integral curve is of degree $p$: the derivative of a degree-$p$ rational curve is of degree $2p$.
- Exact integration of rational curves is hard and requires (whenever possible) non-rational forms.
- The rational model cannot encompass transcendental curves: many of them (helix, cicloid, ...) are of interest in applications.
- Parametrization of conic sections does not correspond to natural arc-length parametrization: unevenly spaced points.

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Drawbacks of the rational model

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- The derivative of a degree-$p$ integral curve is of degree $p + 1$: the derivative of a degree-$p$ rational curve is of degree $2p$.
- Exact integration of rational curves is hard and requires (whenever possible) non-rational forms.
- The rational model cannot encompass transcendental curves: many of them (helix, cycloid, ...) are of interest in applications.
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Drawbacks of the rational model

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Alternatives to the rational model
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\[ \mathbb{P}_p \rightarrow \text{B - splines} \rightarrow \text{NURBS} \]
Alternatives to the rational model

- rational model
  \[ \mathbb{P}_p \rightarrow \text{B-splines} \rightarrow \text{NURBS} \]

- alternative
  \[ < 1, t, \cdots, t^{p-3}, t^{p-2}, t^{p-1} > \]
  \[
  \downarrow \\
  < 1, t, \cdots, t^{p-3}, u(t), v(t) > =: \mathbb{P}^{u,v}_p, \ p \geq 4
  \]
Alternatives to the rational model

- rational model
  \[ \mathbb{P}_p \rightarrow B - splines \rightarrow NURBS \]

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- select \( \mathbb{P}^{u,v}_p \) to exactly represent salient profiles
Alternatives to the rational model

- rational model
  \[ \mathbb{P}_p \rightarrow B - \text{splines} \rightarrow \text{NURBS} \]

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  \[ \downarrow \]
  \[ < 1, t, \cdots, t^{p-3}, u(t), v(t) > =: \mathbb{P}_{u,v}^p, p \geq 4 \]
  - select \( \mathbb{P}_{u,v}^p \) to exactly represent salient profiles
  - construct a basis of \( \mathbb{P}_{u,v}^p \) analogous to Bernstein polynomials (B-basis)
Alternatives to the rational model

\[ \mathbb{P}_p \rightarrow \text{B-splines} \rightarrow \text{NURBS} \]

- alternative

\[ < 1, t, \cdots, t^{p-3}, t^{p-2}, t^{p-1} > \]

\[ \downarrow \]

\[ < 1, t, \cdots, t^{p-3}, u(t), v(t) > =: \mathbb{P}_p^{u,v}, \ p \geq 4 \]

- select \( \mathbb{P}_p^{u,v} \) to exactly represent salient profiles
- construct a basis of \( \mathbb{P}_p^{u,v} \) analogous to Bernstein polynomials (B-basis)
- construct spline spaces with sections in \( \mathbb{P}_p^{u,v} \) with suitable bases for them (analogous to B-splines)
Alternatives to the rational model...

natural choices

\[ \mathbb{P}^u,v_p := \langle 1, t, \cdots, t^{p-3}, \cos \omega t, \sin \omega t \rangle, \]

\[ \mathbb{P}^u,v_p := \langle 1, t, \cdots, t^{p-3}, \cosh \omega t, \sinh \omega t \rangle, \]

conic sections, helix, cicloid ...
Alternatives to the rational model...

\[ \mathbb{P}_p^{u,v} := \langle 1, t, \cdots, t^{p-3}, u(t), v(t) \rangle, \quad p \geq 3 \quad t \in [0, 1] \]
Alternatives to the rational model...

\[ \mathbb{P}_{p}^{u,v} := \langle 1, t, \cdots, t^{p-3}, u(t), v(t) \rangle, \quad p \geq 3 \quad t \in [0, 1] \]

\[ < u^{(p-2)}, v^{(p-2)} > \text{Tchebycheff in } [0, 1] \quad (\forall \text{ element } \neq 0 \text{ has at most 1 zero}) \]
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\[ \Downarrow \]

\[ \mathbb{P}^{u,v}_p \text{ has Bernstein-like basis (B-basis=optimal)} \]
Alternatives to the rational model...

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\[ \langle u^{(p-2)}, v^{(p-2)} \rangle > \text{Tchebycheff in } [0, 1] \quad (\forall \text{ element} \neq 0 \text{ has at most 1 zero}) \]

\[ \mathbb{P}_{p}^{u,v} \text{ has Bernstein-like basis (B-basis=optimal)} \]

Ex:

- \((u, v) = (\cos \omega t, \sin \omega t), \quad 0 < \omega < \pi\)
- \((u, v) = (\cosh \omega t, \sinh \omega t), \quad 0 < \omega\)
- ....
Alternatives to the rational model...

\[ \mathbb{P}^{u,v}_p := \langle 1, t, \cdots, t^{p-3}, u(t), v(t) \rangle, \quad p \geq 3 \quad t \in [0, 1] \]

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\[ \mathbb{P}_{p}^{u,v} := \langle 1, t, \cdots, t^{p-3}, u(t), v(t) \rangle, \quad p \geq 3 \quad t \in [0, 1] \]

\[ \langle u^{(p-2)}, v^{(p-2)} \rangle \text{ Tchebycheff in } [0, 1] \quad (\forall \text{ element } \neq 0 \text{ has at most 1 zero}) \]

\[ \mathbb{P}_{p}^{u,v} \text{ has Bernstein-like basis (B-basis=optimal)} \]

\[ \text{Ex:} \]

1. \((u, v) = (\cos \omega t, \sin \omega t), \quad 0 < \omega < \pi\)
2. \((u, v) = (\cosh \omega t, \sinh \omega t), \quad 0 < \omega\)
\( \mathbb{P}_{p}^{u,v} : \text{Bernstein-like basis} \)

\[
\mathbb{P}_{p}^{u,v} := < 1, t, \cdots, t^{p-3}, u(t), v(t) >, \quad p \geq 3 \quad t \in [0, 1]
\]
\[ \mathbb{P}_{p}^{u,v} : \text{Bernstein-like basis} \]

\[ \mathbb{P}_{p}^{u,v} := < 1, t, \ldots, t^{p-3}, u(t), v(t) >, \ p \geq 3 \ t \in [0, 1] \]

- recurrence relation
\( \mathcal{P}_{p}^{u,v} : \text{Bernstein-like basis} \)

\[
\mathcal{P}_{p}^{u,v} := \langle 1, t, \ldots, t^{p-3}, u(t), v(t) \rangle, \quad p \geq 3 \quad t \in [0, 1]
\]

- recurrence relation
  - \( B_{1,u,v}^{(2)}, B_{2,u,v}^{(2)} \) unique elements in \( \langle u^{(p-2)}, v^{(p-2)} \rangle \).

\[
B_{1,u,v}^{(2)}(0) = 1, \quad B_{1,u,v}^{(2)}(1) = 0, \quad B_{1,u,v}^{(2)}(0) = 0, \quad B_{1,u,v}^{(2)}(1)
\]

- \( \delta_{i,u,v}^{(p)} := \int_{0}^{1} B_{i,u,v}^{(p)}(t)dt \).

\[
B_{i,u,v}^{(p)}(t) := \int_{0}^{t} B_{i-1,u,v}^{(p-1)}(t)ds/\delta_{i-1,u,v}^{(p-1)} - \int_{0}^{t} B_{i,u,v}^{(p-1)}(t)ds/\delta_{i,u,v}^{(p-1)}
\]
I_{u,v}^p : Bernstein-like basis

$I_{u,v}^p := \langle 1, t, \ldots, t^{p-3}, u(t), v(t) \rangle, \quad p \geq 3 \quad t \in [0, 1]$

- recurrence relation
  - $B^{(2)}_{1,u,v}, B^{(2)}_{2,u,v}$ unique elements in $< u^{(p-2)}, v^{(p-2)} >$
  - $B^{(2)}_{1,u,v}(0) = 1, B^{(2)}_{1,u,v}(1) = 0, \quad B^{(2)}_{1,u,v}(0) = 0, B^{(2)}_{1,u,v}(1)$

- $\delta^{(p)}_{i,u,v} := \int_0^1 B^{(p)}_{i,u,v}(t) dt$

$$B^{(p)}_{i,u,v}(t) := \int_0^t B^{(p-1)}_{i-1,u,v}(s) \frac{ds}{\delta^{(p-1)}_{i-1,u,v} - \int_0^t B^{(p-1)}_{i-1,u,v}(t) dt} - \int_0^t B^{(p-1)}_{i,u,v}(t) ds / \delta^{(p-1)}_{i,u,v}$$

Bernstein p.

$$B^{(p)}_i(t) = \left[ \frac{\int_0^t B^{(p-1)}_{i-1}(s) ds}{\int_0^1 B^{(p-1)}_{i-1}(t) dt} - \frac{\int_0^t B^{(p-1)}_{i}(s) ds}{\int_0^1 B^{(p-1)}_{i}(t) dt} \right]$$
\( \mathbb{P}_{p}^{u, v} \) : Bernstein-like basis

\[
\mathbb{P}_{p}^{u, v} := \langle 1, t, \cdots, t^{p-3}, u(t), v(t) >, \quad p \geq 3 \quad t \in [0, 1]
\]

**recurrence relation**

- \( B_{1, u, v}^{(2)} , B_{2, u, v}^{(2)} \) unique elements in \( \langle u^{(p-2)}, v^{(p-2)} \rangle \).

\[
B_{1, u, v}^{(2)} (0) = 1, \quad B_{1, u, v}^{(2)} (1) = 0, \quad B_{1, u, v}^{(2)} (0) = 0, \quad B_{1, u, v}^{(2)} (1)
\]

**degree raising**

\[
\mathbb{P}_{p}^{u, v} \subset \mathbb{P}_{p+1}^{u, v} \Rightarrow \sum_{i=1}^{p} \mathbf{p}_{i} B_{i, u, v}^{(p)} (t) = \sum_{i=1}^{p+1} \hat{\mathbf{p}}_{i} B_{i, u, v}^{(p+1)} (t)
\]

\[
\hat{\mathbf{p}}_{i} := \theta_{i} \mathbf{p}_{i} + (1 - \theta_{i}) \mathbf{p}_{i-1}, \quad \theta_{i} := \frac{D_{i, u, v}^{(p)} (0)}{D_{i, u, v}^{(p+1)} (0)}
\]
(Generalized) B-splines

\[ \Xi := \{ \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p} \}, \quad \Omega := \{ \omega_i, \ldots \}, \quad \text{TRIG.} \quad \omega_i < \frac{\pi}{\xi_{i+1} - \xi_i} \]

\[ B_{i,\Omega,\Xi}^{(2)}(t) := \begin{cases} \frac{\sin \omega_i(t - \xi_i)}{\sin \omega_i(\xi_{i+1} - \xi_i)} & \text{if } t \in [\xi_i, \xi_{i+1}) \\ \frac{\sin \omega_i(\xi_{i+2} - t)}{\sin \omega_i(\xi_{i+2} - \xi_{i+1})} & \text{if } t \in [\xi_{i+1}, \xi_{i+2}) \\ 0 & \text{elsewhere} \end{cases} \]

\[ B_{i,\Omega,\Xi}^{(p)}(t) = \int_{-\infty}^{t} \delta_{i,\Omega,\Xi}^{(p-1)} B_{i,\Omega,\Xi}^{(p-1)}(s)ds - \int_{-\infty}^{t} \delta_{i+1,\Omega,\Xi}^{(p-1)} B_{i+1,\Omega,\Xi}^{(p-1)}(s)ds \]

\[ \delta_{i,\Omega,\Xi}^{(p)} := \frac{1}{\int_{-\infty}^{+\infty} B_{i,\Omega,\Xi}^{(p)}(s)ds} \]
(Generalized) B-splines

\[ \Xi := \{ \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p} \}, \quad \Omega := \{ \omega_i, \ldots \}, \quad \text{TRIG.} : \omega_i < \frac{\pi}{\xi_{i+1} - \xi_i} \]

\[ B^{(2)}_{i,\Omega,\Xi}(t) := \begin{cases} \frac{\sin \omega_i(t-\xi_i)}{\sin \omega_i(\xi_{i+1} - \xi_i)} & \text{if } t \in [\xi_i, \xi_{i+1}) \\ \frac{\sin \omega_i(\xi_{i+2} - t)}{\sin \omega_i(\xi_{i+2} - \xi_{i+1})} & \text{if } t \in [\xi_{i+1}, \xi_{i+2}) \\ 0 & \text{elsewhere} \end{cases} \]

\[ B^{(p)}_{i,\Omega,\Xi}(t) = \int_{-\infty}^{t} \delta^{(p-1)}_{i,\Omega,\Xi} B^{(p-1)}_{i,\Omega,\Xi}(s) \, ds - \int_{-\infty}^{t} \delta^{(p-1)}_{i+1,\Omega,\Xi} B^{(p-1)}_{i+1,\Omega,\Xi}(s) \, ds \]

\[ \delta^{(p)}_{i,\Omega,\Xi} := \frac{1}{\int_{-\infty}^{+\infty} B^{(p)}_{i,\Omega,\Xi}(s) \, ds} \]

B-splines

\[ B^{(p)}_{i}(t) = \int_{-\infty}^{t} \delta^{(p-1)}_{i,\Xi} B^{(p-1)}_{i,\Xi}(s) \, ds - \int_{-\infty}^{t} \delta^{(p-1)}_{i+1,\Xi} B^{(p-1)}_{i+1,\Xi}(s) \, ds \]

\[ \delta^{(p)}_{i,\Xi} := \frac{1}{\int_{-\infty}^{+\infty} B^{(p)}_{i,\Xi}(s) \, ds} \]
(Generalized) B-splines

\[ \Xi := \{ \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p} \}, \quad \Omega := \{ \ldots, \omega_i, \ldots \}, \quad \text{TRIG.} : \omega_i < \frac{\pi}{\xi_{i+1} - \xi_i} \]

\[ B_{i,\Omega,\Xi}^{(2)}(t) := \begin{cases} 
  \frac{\sin \omega_i(t-\xi_i)}{\sin \omega_i(\xi_{i+1} - \xi_i)} & \text{if } t \in [\xi_i, \xi_{i+1}) \\
  \frac{\sin \omega_i(\xi_{i+2} - t)}{\sin \omega_i(\xi_{i+2} - \xi_{i+1})} & \text{if } t \in [\xi_{i+1}, \xi_{i+2}) \\
  0 & \text{elsewhere} 
\end{cases} \]

\[ B_{i,\Omega,\Xi}^{(p)}(t) = \int_{-\infty}^{t} \delta_{i,\Omega,\Xi}^{(p-1)} B_{i,\Omega,\Xi}^{(p-1)}(s)ds - \int_{-\infty}^{t} \delta_{i+1,\Omega,\Xi}^{(p-1)} B_{i+1,\Omega,\Xi}^{(p-1)}(s)ds \]

\[ \delta_{i,\Omega,\Xi}^{(p)} := \frac{1}{\int_{-\infty}^{+\infty} B_{i,\Omega,\Xi}^{(p)}(s)ds} \]
(Generalized) B-splines ...

\[ \{ B_{i,\Omega,\Xi}^{(p)}(t), \ i = 1, \cdots \}, \]

- properties
(Generalized) B-splines ...

\[ \{ B_{i,\Omega,\Xi}^{(p)}(t), \ i = 1, \ldots \}, \]

- properties
  - positivity
  - p. of unity
  - compact support
  - smoothness
  - derivative
  - local linear independence
  - ...

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(Generalized) B-splines ...

\{ B^{(p)}_{i,\Omega,\Xi}(t), \ i = 1, \ldots \},

- properties
  - positivity
  - p. of unity
  - compact support
  - smoothness
  - derivative
  - local linear independence
  - ...

- shape properties \{\ldots, \omega_i, \ldots\}
(Generalized) B-splines ...

\( \{ B_i^{(p)}_{\Omega, \Xi}(t), \ i = 1, \cdots \} \),

- properties
  - positivity
  - p. of unity
  - compact support
  - smoothness
  - derivative
  - local linear independence
  - ...
- shape properties \( \{ \cdots, \omega_i, \cdots \} \)
(Generalized) B-splines ...

\[ \{ B_{i,\Omega,\Xi}^{(p)}(t), \ i = 1, \cdots \} , \]

- properties
  - positivity
  - p. of unity
  - compact support
  - smoothness
  - derivative
  - local linear independence
  - ...
- shape properties \{\ldots, \omega_i, \ldots\}
$(Generalized)$ B-splines...

\[ \{ B_{i,\Omega,\Xi}^{(p)}(t), \ i = 1, \ldots \}, \]

\[ \omega_i \to 0 \Rightarrow B_{i,\Omega,\Xi}^{(p)}(t) \to B_{i,\Xi}^{(p)}(t) \]
(Generalized) B-splines ...

\[ \{ B_{i,\Omega,\Xi}^{(p)}(t), \ i = 1, \cdots \} \],

- \( \omega_i \rightarrow 0 \Rightarrow B_{i,\Omega,\Xi}^{(p)}(t) \rightarrow B_{i,\Xi}^{(p)}(t) \)

- unified treatment by complex arithmetic
(Generalized) B-splines ...

\[
\{ B_{i,\Omega,\Xi}^{(p)}(t), \ i = 1, \cdots \},
\]

- \( \omega_i \to 0 \Rightarrow B_{i,\Omega,\Xi}^{(p)}(t) \to B_{i,\Xi}^{(p)}(t) \)
- unified treatment by complex arithmetic
- trig. and exponent. parts can be mixed
(Generalized) B-splines: knot insertion

\[ \Xi := \{ \cdots \leq \xi_i < \xi_{i+1} \leq \cdots \} \subset \{ \cdots \leq \xi_i \leq \bar{\xi} < \xi_{i+1} \leq \cdots \} =: \bar{\Xi}, \quad S_{\Xi,\Omega,p} := \sum_j c_j B_{j,\Xi,\Omega}^{(p)} \]
(Generalized) B-splines: knot insertion

\[ \Xi := \{ \cdots \leq \xi_i < \xi_{i+1} \leq \cdots \} \subset \{ \cdots \leq \xi_i \leq \xi_{i+1} \leq \cdots \} =: \Xi, \quad S_{\Xi,\Omega,p} := \{ \sum_j c_j B_{j,\Xi,\Omega}^{(p)} \} \]

\[ \sum_j c_j B_{j,\Omega,\Xi}^{(p)} = \sum_j \tilde{c}_j B_{j,\Omega,\Xi}^{(p)} \]

\[ \tilde{c}_j = \gamma_{j,p} c_j + (1 - \gamma_{j,p}) c_{j-1} \]

\[ \gamma_{j,p} := \begin{cases} 1 & j \leq i - p \\ \frac{\delta_j^{(p-1)}(\Xi,\Omega,\Xi)}{\delta_j^{(p-1)}(\Xi,\Omega,\Xi)} \gamma_j, p-1 & i - p < j < i - r + 1 \\ 0 & j \geq i - r + 1 \end{cases} \quad \gamma_j,2 := \begin{cases} 1 & j \leq i \\ \frac{\sin \omega_i (\xi_{i+1} - \xi_i)}{\sin \omega_i (\xi_{i+1} - \xi_i)} & j = i \\ 0 & j \geq i + 1 \end{cases} \]

r mult. of \( \tilde{\xi} \) in \( \Xi \)
Tensor-product (Generalized) B-splines surfaces

$$\Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p}\}, \ \Upsilon := \{\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{m+q}\}, \ \Omega := \{\ldots \omega_i \ldots\}, \ \Lambda := \{\ldots \lambda_j \ldots\}$$

$$S(s, t) = \sum_{i=1, j=1}^{n,m} c_{i,j} B_{i,\Omega,\Xi}^{(p)}(s) B_{i,\Lambda,\Upsilon}^{(q)}(t)$$
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S(s, t) = \sum_{i=1, j=1}^{n,m} c_{i,j} B_{i,\Xi}(s) B_{i,\Upsilon}(t)
\]

\((s \cos(t), s \sin(t), t)\) helicoid
Conclusion

NURBS are a powerful tool in Isogeometric analysis due to their ability to exactly represent profiles of interest in applications and to the wealth of efficient algorithms for their manipulation.
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NURBS are not a requisite ingredient in Isogeometric analysis

[Huges-Cottrel-Bazilevs, 2005]
Conclusion

- NURBS are a powerful tool in Isogeometric analysis due to their ability to exactly represent profiles of interest in applications and to the wealth of efficient algorithms for their manipulation.

- *NURBS are not a requisite ingredient in Isogeometric analysis*  
  [Hughes-Cottrel-Bazilevs, 2005]

- Generalized (trigonometric/hyperbolic) B-splines could offer a possible alternative to the rational model.
References: Bernstein p. & B-splines


MATLAB: Spline Toolbox

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Quasi-interpolation

quasi-interpolation: general approach to construct, with low computational cost, efficient local approximants to a given data set/function
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\[ f \rightarrow \sum_i \lambda_i(f) \varphi_i \]

\[ \{ \varphi_1, \cdots, \varphi_m \} \text{ positive p.unity } \rightarrow \text{ stability,} \]

\[ \{ \varphi_1, \cdots, \varphi_m \} \text{ local support } \rightarrow \text{ local control} \]

\[ \lambda_i(f) : \text{ depends on } f \text{ and on its derivatives/integrals} \]
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- q.i. based on B-splines/NURBS/Generalized B-splines are widely studied
- q.i. could be a useful tool in Isogeometric analysis (boundary conditions, error estimates, ...)

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